Tutorial 3

Two-person zero-sum games

Definition 1. *A game is called a* two-person zero-sum game *if*

(i) Two players make their moves simultaneously.

(ii) One player wins what the the other player loses.

Strategic form

Definition 2. *A* strategic form *of a two-person zero-sum game is a triple (X, Y, π), where X, Y are the sets of strategies of Player I and Player II respectively, and $\pi : X \times Y \to \mathbb{R}$ is the payoff function of Player I.*

In this note, we only consider the case that both X and Y are finite, so that we can identify the payoff function as a matrix.

Matrix game

Assume $X = \{1, \cdots, m\}$, $Y = \{1, \cdots, n\}$ are the sets of strategies of Player I (the row player) and Player II (the column player) respectively. Let $A \in M_{m\times n}(\mathbb{R})$ be the payoff matrix, that is, $a_{i,j}$ denotes the payoff of the the row player when the row player takes his strategy i and the column player takes his strategy j.

Pure strategy: If A has a saddle point $a_{k,l}$, that is

$$a_{k,l} = \min_{1\leq j\leq n} a_{k,j} = \max_{1\leq i\leq m} a_{i,l},$$

then the row player has an optimal pure strategy k and the column has an optimal pure strategy l.

Mixed strategy: Let $\mathcal{P}^m$ denote the collection of p dimensional probability vectors. We call each probability vector $\boldsymbol{p} \in \mathcal{P}^m$ a mixed strategy for the row player. Similarly, each $\boldsymbol{q} \in \mathcal{P}^n$ is called a mixed strategy for the column player.

Theorem 1. *(Minimax Theorem). Let A be an $m \times n$ matrix. Then there exist a number $v \in \mathbb{R}$ and two probability vectors $\boldsymbol{p} \in \mathcal{P}^m$, $\boldsymbol{q} \in \mathcal{P}^n$ such that*

(i) $\boldsymbol{p}A\boldsymbol{y}^T \geq v$ for any $\boldsymbol{y} \in \mathcal{P}^n$.

(ii) $\boldsymbol{x}A\boldsymbol{q}^T \leq v$ for any $\boldsymbol{x} \in \mathcal{P}^m$.

(iii) $\boldsymbol{p}A\boldsymbol{q}^T = v$.

Remark: (1) The number v in the above theorem is unique, and we call it the value of A, write $v = v(A)$.

(2) In the above theorem, we call $\boldsymbol{p}$ an optimal (mixed) strategy for the row player and $\boldsymbol{q}$ an optimal (mixed) strategy for the column player. In general, $\boldsymbol{p}$ and $\boldsymbol{q}$ may not be unique.

(3) If $v = 0$, we say this game is fair.

(4) By solving a matrix game, we mean finding the value of matrix A and optimal strategies for the two players.

Exercise 1. *Show that the number v in the Minimax Theorem is unique.*

Proof. Suppose two triples $(v, \boldsymbol{p}, \boldsymbol{q})$, $(v', \boldsymbol{p}', \boldsymbol{q}')$ both satisfy (i), (ii), (iii) in the Minimax Theorem. Note that by using (i), (ii) several times, we have

$$v \leq \boldsymbol{p}A\boldsymbol{q}'^T \leq v' \leq \boldsymbol{p}'A\boldsymbol{q}^T \leq v.$$

Exercise 2. *Prove if $A^T = -A$, then $v(A) = 0$.*

Proof. Write $v(A) = v$. Assume $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{P}^n$ are optimal strategies. Then by the Minimax Theorem, we have

$$\begin{cases} \boldsymbol{p}A\boldsymbol{y}^T \geq v, & \forall \boldsymbol{y} \in \mathcal{P}^n. \\ \boldsymbol{x}A\boldsymbol{q}^T \leq v, & \forall \boldsymbol{x} \in \mathcal{P}^n. \\ \boldsymbol{p}A\boldsymbol{q}^T = v. \end{cases}$$

Taking transpose in the above equations and applying the assumption that $A^T = -A$, we have

$$\begin{cases} \boldsymbol{y}A\boldsymbol{p}^T \leq -v, & \forall \boldsymbol{y} \in \mathcal{P}^n. \\ \boldsymbol{q}A\boldsymbol{x}^T \geq -v, & \forall \boldsymbol{x} \in \mathcal{P}^n. \\ \boldsymbol{q}A\boldsymbol{p}^T = -v. \end{cases}$$

By the Minimax Theorem and the uniqueness of the value of A, we have $v = -v$, hence $v = 0$.

Solving matrix games

Two useful principles: 1. Deleting the dominated rows and columns to obtain a new matrix with lower dimensions. Recall that a row is dominated if it is dominated (or say bounded) from above by another row, a column is dominated if it is dominated from below by another column.

2. The principle of indifference. Assume $\boldsymbol{p} = (p_1, \cdots, p_m)$ and $\boldsymbol{q} = (q_1, \cdots, q_n)$ are optimal strategies for Player I and Player II respectively. Then

(i) for any $k \in \{1, \cdots, m\}$ with $p_k > 0$, we have $\sum_{j=1}^n a_{k,j} q_j = v(A)$.

(ii) for any $l \in \{1, \cdots, n\}$ with $q_l > 0$, we have $\sum_{i=1}^m a_{i,l} p_i = v(A)$.

Exercise 3. *In a Rock-Paper-Scissors game, the loser pays the winner an amount of money which is equal to the total number of fingers shown by the two players (for example, if Player I shows Scissors and Player II shows Paper, then Player II should pay* 7 *dollars to Player I).*

(i) Find the value of the games.

(ii) Find optimal strategies for the two players.

Solution. The game is clearly a two-person zero-sum game and the game matrix is given by

$$A = \begin{array}{c} \\ R \\ P \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & P & S \end{array} \\ \begin{pmatrix} 0 & -5 & 2 \\ 5 & 0 & -7 \\ -2 & 7 & 0 \end{pmatrix} \end{array}.$$

(i) Since $A^T = -A$, we have $v(A) = 0$.

(ii) Assume $\boldsymbol{q} = (q_1, q_2, q_3)$ is an optimal strategy for Player I. Assume q_1, q_2, q_3 are all positive, then by the principle of indifference, we have

$$\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} 0 & -5 & 2 \\ 5 & 0 & -7 \\ -2 & 7 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}.$$

Hence we have

$$\begin{cases} 5p_2 - 2p_3 = 0 \\ -5p_1 + 7p_3 = 0 \\ 2p_1 - 7p_2 = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

Solving the above equations, we get $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{7}$, $p_3 = \frac{5}{14}$. Similarly, assume $\boldsymbol{p} = (p_1, p_2, p_3)$ is an optimal strategy for Player II and $\boldsymbol{p}$ is strictly positive, we have $\boldsymbol{q} = (\frac{1}{2}, \frac{1}{7}, \frac{5}{14})$. It is easy to check $v = 0$, $\boldsymbol{p} = \boldsymbol{q} = (\frac{1}{2}, \frac{1}{7}, \frac{5}{14})$ satisfy the the conclusion of the Minimax Theorem. Hence $v = 0$ is the value of A and $\boldsymbol{p} = \boldsymbol{q} = (\frac{1}{2}, \frac{1}{7}, \frac{5}{14})$ are optimal strategies.

Exercise 4. *Let*

$$A = \begin{pmatrix} 0 & -2 & 2 & 1 & 4 \\ 2 & -1 & 3 & 0 & 5 \\ 3 & 4 & -2 & 5 & -3 \end{pmatrix}$$

(i) Find the reduced matrix of A by deleting dominated rows and columns.

(ii) Solve the two-person zero-sum game with game matrix A.

Solution. (i) Note that the fourth column is dominated by the second column from below, by deleting the fourth column we obtain

$$\begin{pmatrix} 0 & -2 & 2 & 4 \\ 2 & -1 & 3 & 5 \\ 3 & 4 & -2 & -3 \end{pmatrix}.$$

Now the first row in dominated by the second row from above, by deleting the first row we obtain

$$\begin{pmatrix} 2 & -1 & 3 & 5 \\ 3 & 4 & -2 & -3 \end{pmatrix}.$$

There are no more dominated rows or columns, hence the above matrix is the desired reduced matrix.

(ii) Let A' denote the reduced matrix. For $x \in [0,1]$, we have

$$(x, 1-x)A' = (2x + 3(1-x), -x + 4(1-x), 3x - 2(1-x), 5x - 3(1-x)).$$

Draw the graph of

$$\begin{cases} C_1 : v = 2x + 3(1-x) = 3 - x \\ C_2 : v = -x + 4(1-x) = 4 - 5x \\ C_3 : v = 3x - 2(1-x) = 5x - 2 \\ C_5 : v = 5x - 3(1-x) = 8x - 3 \end{cases}.$$

The lower envelope is shown in Figure 1. Solving

$$\begin{cases} C2 : v = 4 - 5x \\ C3 : v = 5x - 2 \end{cases},$$

we have $v = 1$ and $x = 0.6$. Hence $v(A) = 1$ and the optimal strategy for the row player is $(0, 0.6, 0.4)$. Solving

$$\begin{cases} R2 : -y + 3(1-y) = 1 \\ R3 : 4y - 2(1-y) = 1 \end{cases},$$

we have $y = 0.5$. Hence the optimal strategy for the column player is $(0, 0.5, 0.5, 0, 0)$.

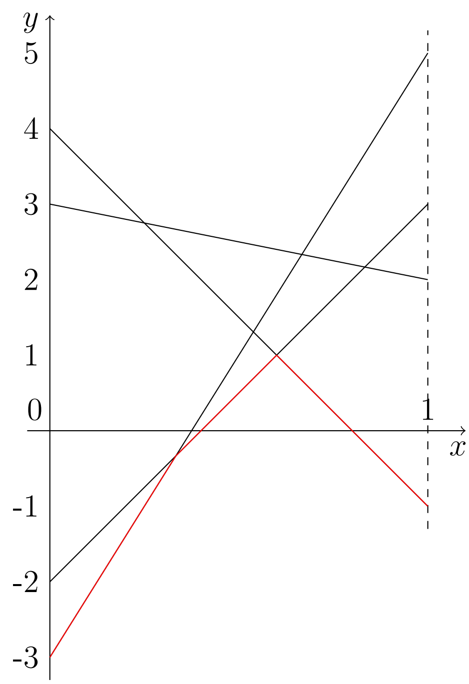

Figure 1